Direct Sums of Ideals

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ABSTRACT

We survey various existence and uniqueness theorems for decompositions of finitely generated modules over commutative rings, as direct sums of ideals of the ring. These theorems generalize a theorem of Steinitz published in 1912. Much of the paper is expository. The main new result is the following uniqueness theorem (well known to be true for integral domains): Let A_i, B_i be ideals of the commutative ring R, and suppose that the R-modules $A_1 \oplus \cdots \oplus A_m$ and $B_1 \oplus \cdots \oplus B_m$ are isomorphic. Then the ideal products $A_1 \cdots A_m$ and $B_1 \cdots B_m$ are isomorphic as R-modules.

INTRODUCTION

The main new result in this paper is a uniqueness theorem in the theory of direct-sum decompositions of certain modules over commutative rings. To set the stage we recall the fundamental theorem of Abelian groups, which states that every finitely generated module over a principal ideal domain D is isomorphic to a direct sum $D/(d_1) \oplus \cdots \oplus D/(d_m)$, where the ideals (d_i) satisfy the inclusions $D \supset (d_1) \supseteq \cdots \supseteq (d_m)$. Moreover, the integer m and the ideals (d_i) are uniquely determined by the isomorphism class of the module. If the module is assumed to be torsion-free, it follows that $d_i = 0$ for all i, and we recover the well-known fact that every finitely generated torsion-free D-module is free.

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In this paper we will survey various generalizations of this last assertion, starting with a famous result published in 1912 by Steinitz [26]. His theorem was stated for the ring of integers in an algebraic number field, but it can be viewed as a structure theorem for finitely generated torsion-free modules over Dedekind domains.

1. DEDEKIND DOMAINS; STEINITZ'S THEOREM

A Dedekind domain is an integral domain D whose isomorphism classes of nonzero ideals form a group under the operation $[A][B]=[AB]$, where $[A]$ denotes the isomorphism class of the ideal A, and AB is the ideal consisting of finite sums of elements of the form ab with $a \in A$ and $b \in B$. (An isomorphism between ideals is just an isomorphism as D-modules.) Steinitz's theorem runs as follows:

THEOREM 1 (Steinitz, 1912). *Let D be a Dedekind domain, and let M be a nonzero, finitely generated, torsion-free D-module. Then $M \cong A_1 \oplus \cdots \oplus A_m$, where the A_i are nonzero ideals of D. Moreover, the integer m and the isomorphism class of the product $A_1 \cdots A_n$ form a complete set of invariants for the isomorphism class of the module M.*

This theorem really contains three assertions:

(1.1) an existence theorem, saying that each finitely generated torsion-free D-module M admits a decomposition as a direct sum of ideals;

(1.2) a weak uniqueness theorem, which says that if A_i and B_j are nonzero ideals of D such that $A_1 \oplus \cdots \oplus A_m$ and $B_1 \oplus \cdots \oplus B_n$ are isomorphic, then $m=n$ and the products $A_1 \cdots A_m$ and $B_1 \cdots B_m$ are isomorphic; and

(1.3) a *nonuniqueness* theorem, saying that if A_i, B_i are nonzero ideals of D, $1 \leqslant i \leq m$, and if $A_1 \cdots A_m \cong B_1 \cdots B_m$, then $A_1 \oplus \cdots \oplus A_m$ and $B_1 \oplus \cdots \oplus B_m$ are isomorphic.

We call (1.3) a nonuniqueness theorem because it implies that a module can have more than one decomposition as a direct sum of ideals. For example, we have $D \oplus D \cong A \oplus A^{-1}$ for any nonzero ideal A of the Dedekind domain D. (By a harmless abuse of notation, we write A^{-1} for any ideal B satisfying $[B]=[A]^{-1}$.) More generally, we have $U \oplus V \cong D \oplus UV$ for any two ideals U, V of D. Since $A \cong D$ only if A is principal, we see that *true* uniqueness (up to isomorphism and rearrangement of the summands) never holds for direct sums of ideals over a Dedekind domain that is not a principal-ideal domain. (Well-known examples of nonprincipal Dedekind domains include $\mathbf{Z}[\sqrt{-5}]$, $\mathbf{R}[x,y]/(x^2+y^2-1)$, and $\mathbf{C}[x,y]/(y^2-x^3-x)$.)

Notice that Steinitz's theorem yields a canonical form for finitely generated torsion-free modules over Dedekind domains: Given an arbitrary decomposition $M \cong A_1 \oplus \cdots \oplus A_m$, put $A = A_1 \cdots A_m$. By (1.3), $M \cong D^{(m-1)} \oplus A$, and by (1.2) the ideal A is determined up to isomorphism by M. (We always use a parenthetical exponent for the direct sum of copies of a module. Exponents without parentheses are used for powers of an ideal; e.g., $A^2 = AA$.)

We will survey what is known about possible extensions of these three theorems to more general commutative rings. We will see that (1.1) and (1.3) impose severe restrictions on the ring, but that (1.2) holds for all commutative rings, as long as we *assume* that $m = n$.

2. PRÜFER DOMAINS; KAPLANSKY'S THEOREM

A natural way to generalize Dedekind domains is to allow the ring to be non-Noetherian, but to have well-behaved finitely generated ideals. One defines an ideal A in a ring R to be invertible provided there is an ideal B such that $AB \cong R$; equivalently, AB is a principal ideal generated by a non-zero-divisor. As before, we write A^{-1} for any such ideal B. Invertible ideals are always finitely generated. A domain in which every finitely generated nonzero ideal is invertible is said to be a Prüfer domain. An interesting example of a Prüfer domain is the ring E of integer-valued polynomials studied by Pólya [22] and Ostrowski [21] in 1919. (An integer-valued polynomial is a polynomial $f(x) \in \mathbf{Q}[x]$ with the property that $f(n) \in \mathbf{Z}$ for each $n \in \mathbf{Z}$. For example, take $f(x) = x(x+1)/2$.) The fact that E is Prüfer is proved in [6] and [5].

Here is Kaplansky's non-Noetherian analog of Steinitz's theorem, from [17]:

THEOREM 2 (Kaplansky, 1952). *Let D be a Prüfer domain, and let M be a finitely generated torsion-free D-module. Then $M \cong A_1 \oplus \cdots \oplus A_m$ for suitable nonzero ideals $A_1, \ldots, A_m$. Moreover, the integer m and the isomorphism class of the product $A_1 \cdots A_m$ are uniquely determined by M.*

Kaplansky actually proved the uniqueness part of his theorem for arbitrary nonzero ideals of any integral domain. We will give his beautiful proof here, as a warmup for our proof of the general form of (1.2).

THEOREM 3 (Kaplansky, 1952). *Let A_i and B_j be ideals of a commutative ring R, each containing a non-zero-divisor. If $A_1 \oplus \cdots \oplus A_m \cong B_1 \oplus \cdots \oplus B_n$, then $m = n$ and $A_1 \cdots A_m \cong B_1 \cdots B_m$.*

Proof. It is convenient to use the total quotient ring S, which consists of formal fractions r/s, where $r, s \in R$ and s is a non-zero-divisor of R. To see that $m = n$, we note that $A_i S = S$ and $B_j S = S$ for all i, j, since every non-zero-divisor of R is a unit of S. The hypothesized isomorphism induces an isomorphism of S-modules $S^{(m)} \cong S^{(n)}$. Now let $\mathfrak{m}$ be any maximal ideal of S, and pass to the field $k = S/\mathfrak{m}$, getting $k^{(m)} \cong k^{(n)}$ as vector spaces, whence $m = n$.

Let $s_i \in A_i$ and $t_i \in B_i$ be the promised non-zero-divisors, and let $U_i = s_i^{-1}A_i$ and $V_i = t_i^{-1}B_i$. These are R-submodules of S and are isomorphic to A_i and B_i, respectively. It will suffice to prove that $U_1 \cdots U_m$ and $V_1 \cdots V_m$ are isomorphic. Choose an isomorphism $\Phi : U_1 \oplus \cdots \oplus U_m \to V_1 \oplus \cdots \oplus V_n$, and let Ψ be its inverse. Then Φ is given by a matrix of maps $\Phi_{ij} : U_j \to V_i$, and we put $f_{ij} = \Phi_{ij}(1)$. Note that Φ_{ij} is multiplication by f_{ij}. For, if $x \in U_j$ then $s_j x \in A_j$, and we have $s_j \Phi_{ij}(x) = \Phi_{ij}(s_j x) = s_j x \Phi_{ij}(1) = s_j x f_{ij}$; now cancel the non-zero-divisor s_j. It follows that Φ is left multiplication by the matrix $\phi := [f_{ij}]$. (Write m-tuples as column vectors.) Similarly, let $\Psi_{jk} : V_k \to U_j$ be the components of Ψ, and let $g_{jk} = \Psi_{jk}(1)$. Then Ψ is left-multiplication by $\psi := [g_{jk}]$.

Now we consider the determinants $\delta := \det \phi$ and $\varepsilon := \det \psi$. Since the matrices ϕ and ψ are inverses of each other, we have $\delta\varepsilon = 1$. One checks directly (or elegantly, using the product rule for determinants) that multiplication by δ carries the product $U_1 \cdots U_m$ into $V_1 \cdots V_m$. Combining this fact with the symmetric assertion about ε, we obtain reciprocal isomorphisms between $U_1 \cdots U_m$ and $V_1 \cdots V_m$. ■

Notice that the nonuniqueness assertion does not appear in Kaplansky's theorem. By the following theorem, the finitely generated ideals of any Prüfer domain satisfying (1.3) have to be generated by two elements. This theorem also throws a wet blanket on any attempt at a literal generalization of part (1.3) of Steinitz's theorem within the context of Noetherian rings.

THEOREM 4. *Let A be an ideal of a ring R, and assume A contains a non-zero-divisor. The following are equivalent:*

(a) $A \oplus A \cong R \oplus W$ *for some R-module W.*
(b) $A \oplus A \cong R \oplus A^2$.
(c) *A is invertible and* $A \oplus A^{-1} \cong R \oplus R$.
(d) *A is invertible and is generated by two elements.*

Proof. If (c) holds, we multiply both sides of the isomorphism by A to get (b). Obviously (b) implies (a). Assuming (a), we will prove (d). Let s be a non-zero-divisor in A. Then R and consequently sR are homomorphic images of $A \oplus A$. Therefore we have R-homomorphisms $f, g : A \to sR$ and elements $u, v \in A$ such that $f(u) + g(v) = s$. Write $f(s) = xs$ and $g(s) = ys$. As in the proof of Theorem 3, we see that

$$f(a) = ax \text{ and } g(a) = ay \qquad \text{for all} \quad a \in A, \tag{2.1}$$

and in particular

$$ux + vy = s. \tag{2.2}$$

It follows from (2.1) and (2.2) that $AB = sR$, where $B = xR + yR$. Thus A is invertible. To complete the proof of (d), we show that u and v generate A, as follows: Given $a \in A$, (2.2) and (2.1) yield $sa = uax + vay = uf(a) + vg(a) \in usR + vsR$. Now cancel s to get $a \in uR + vR$, as desired.

Next we show that (d) $\Rightarrow$ (b). Then, on multiplying the isomorphism in (b) by A^{-1}, we'll have (d) $\Rightarrow$ (c), completing the cycle. Let $A = uR + vR$, and suppose $AB = sR$, where B is an ideal and s is a non-zero-divisor. Then $ux + vy = s$ for suitable elements $x, y \in B$. Let $\Phi : A \oplus A \to Rs \oplus A^2$ be defined by the matrix

$$\phi = \begin{bmatrix} x & y \\ -v & u \end{bmatrix}.$$

A routine computation shows that $\ker \Phi = 0$, and that the image contains $\begin{bmatrix} s \\ 0 \end{bmatrix}$ as well as elements with u^2, uv, and v^2 as their second entries. It follows that Φ is surjective, and the proof is complete. ∎

In view of Theorem 4 we will discuss property (1.3) only for invertible ideals. It is easy to see that the validity of (1.3) for all invertible ideals of a ring R is equivalent to the following assertion, which is known as the Steinitz property:

(SP) If A and B are invertible ideals of R, then $A \oplus B$ is isomorphic to $R \oplus AB$.

We say the Prüfer domain has the n-generator property provided every finitely generated ideal is generated by at most n elements. Theorem 4 says that the 2-generator property is equivalent to the weak form of (SP) in which $A = B$, but it is apparently unknown whether these properties are equivalent to (SP) itself.

There is no reason to expect invertible ideals to be 2-generated, just because that is the bound for Dedekind domains. Already in 1968, Gilmer [10] had shown, by a clever application of the Borsuk-Ulam theorem, that for each n there is a Noetherian ring with an invertible ideal requiring n generators. (S. U. Chase had obtained such examples in earlier unpublished work.) For many years the big problem in the theory of Prüfer domains was to determine whether or not every Prüfer domain has the 2-generator property. Although the problem was finally answered negatively in 1979, several positive results related to the n-generator property and (SP) appeared earlier. In studying these problems, one is led rather naturally to consideration of the "$n\frac{1}{2}$-generator" property. One says that an ideal A is $n\frac{1}{2}$-generated provided every nonzero element of A is part of an $(n+1)$-element generating set for A. Dedekind domains—in fact, all one-dimensional Prüfer domains—have the $1\frac{1}{2}$-generator property. In 1975 Heitmann and Levy [16] showed that every Prüfer domain with the $1\frac{1}{2}$-generator property satisfies (SP). In the same paper they constructed an example of a Prüfer domain D without the $1\frac{1}{2}$-generator property. Alas, D has the 2-generator property and satisfies (SP). Later Gilmer and Smith showed, in [12] and [13], that the ring of integer-valued polynomials (which has dimension 2) has the 2-generator property but not the $1\frac{1}{2}$-generator property.

In 1976 Heitmann [14] proved that every Prüfer domain of dimension n has the $n\frac{1}{2}$-generator property. Finally, in 1979, Schülting [24] produced a brilliant example of a Prüfer domain of dimension 2 with an ideal needing three generators, and in 1984 Swan [25] showed that for every n there is a Prüfer domain of dimension n that does not satisfy the n-generator property. (Thus the bound in [14] is sharp.) Swan also produced a Prüfer domain in which the n-generator property fails for every n. As far as we have been able to determine, Schülting's example was the first example of a Prüfer domain without the Steinitz property.

We remark that there are easy examples of ideals A, B, C of a domain R, none of them invertible, such that $A \oplus B \cong R \oplus C$. In fact, this sort of example is really the easiest way to see nonuniqueness of direct-sum decompositions.

EXAMPLE 1. Let $R = k[x, y]$, the polynomial ring in two variables over a field. Let A and B be any two distinct maximal ideals of R, and let $C = AB$. Then none of the ideals A, B, C is invertible, yet $A \oplus B \cong R \oplus C$.

Proof. We have $A + B = R$, so there is a split surjection $\Phi : A \oplus B \to R$ defined by $\Phi(a, b) = a - b$. Therefore $A \oplus B \cong R \oplus \ker \Phi$, and $\ker \Phi = A \cap B$. But $A \cap B = AB$ because $A + B = R$. Since invertible prime ideals have height one (in any Noetherian ring), A and B are not invertible, and therefore neither is their product C. ■

The following example shows that it is possible for an ideal consisting of zero divisors to satisfy (b) of Theorem 4. Since every invertible ideal contains a non-zero-divisor, this shows that (b) and (d) are no longer equivalent without the assumption that A contains a non-zero-divisor.

EXAMPLE 2. There is a ring R with a two-generator maximal ideal A, consisting of zero divisors and satisfying $A \oplus A \cong R \oplus A^2$. Moreover, A is a projective R-module of constant rank one.

Proof. Start with a countable Dedekind domain D having two isomorphic maximal ideals P and Q, and assume that $[P]$ has infinite order in the ideal class group. (Such a domain exists by [7].) The idea is to adjoin to D elements that kill the elements of P in such a way that the extensions of P and Q are still isomorphic. Choose a fixed R-isomorphism $\phi : P \to Q$.

Suppose we have a countable ring S containing D and an ideal L of S such that $PS \supseteq L$, $L_P = 0$, and $S = D \oplus L$ (an internal direct sum as additive groups). Assume there is an S-isomorphism $\psi : PS \to QS$ extending ϕ. Let a be an arbitrary element of PS. Then $a = b + c$, with $b \in P$ and $c \in L$. Note that neither P nor $P \cap Q$ is the radical of the principal ideal Db. (For otherwise we would have $r > 0$, $s \geqslant 0$, such that $Dd = P^r Q^s \cong P^{r+s}$, contradicting the assumption that $[P]$ has infinite order.) Therefore D has a maximal ideal U containing b and distinct from both P and Q. Let Z be an indeterminate, let $T = S[Z]$, and put $S^\wedge = T/Z(U \oplus L)T$. Let z be the image of Z in $S^\wedge$. Then $S^\wedge = S[z] = S \oplus zS^\wedge = D \oplus L^\wedge$, where $L^\wedge = L \oplus zS^\wedge$. Since $P + U = D$, it follows that $z \in PS^\wedge$, and we have $PS^\wedge \supseteq L^\wedge$. Also, $L^\wedge{}_P = 0$, because $zU = 0$. Now $za = 0$, and we want to extend ψ to an $S^\wedge$-isomorphism $\psi^\wedge$ from $PS^\wedge$ to $QS^\wedge$. It is helpful to think of $S^\wedge$ as a direct sum: $S^\wedge = S \oplus \bar{S}z \oplus \bar{S}z^2 \oplus \cdots$, where $\bar{S} = S/(U \oplus L) \cong D/U$. Now $PS^\wedge = PS \oplus \bar{S}z \oplus \bar{S}z^2 \oplus \cdots$, since $P + U = D$; and $QS^\wedge$ has a similar decomposition. We define the extension $\psi^\wedge$ on each summand, using ψ in degree 0. In higher degrees, we note that $\bar{S} = (P + U)/U = P/PU$, and ϕ carries this quotient isomorphically onto $Q/QU = \bar{S}$, thereby inducing an automorphism of $\bar{S}z^i$. Clearly $\psi^\wedge$ is an S-isomorphism from $PS^\wedge$ onto $QS^\wedge$, and since it commutes with multiplication by z, it is an $S^\wedge$-isomorphism.

Put $R_0 = D$, and build extensions R_i inductively as follows: Enumerate the elements of PR_i. Using the construction above a countable number of times, we obtain a ring $R_{i+1} = D \oplus L_i$ containing R_i such that every element of PR_i is a zero divisor in R_{i+1}, and an R_{i+1}-isomorphism from PR_{i+1} to QR_{i+1} extending ϕ. Let R be the union of the R_i's, and put $A = PR$. Then A consists of zero divisors, and $A \cong B := QR$. Then $A \oplus A \cong A \oplus B \cong R \oplus A^2$, exactly as in Example 1.

At each stage of the construction we have $PS \supseteq L$, whence $S = D + PS$. Therefore $R = D + A$, and it follows that $R/A \cong D/P$. Therefore A is a maximal ideal of R. To show that A is projective, it is enough to prove that $A_{\mathfrak{m}} \cong R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of A. (See [4, II, §5, Théorème 2, p. 141].) We may assume that $\mathfrak{m} = A$, since A blows up at every other maximal ideal. Now $L_P = 0$ at each stage, so $R_{\mathfrak{m}} \cong D_P$, a discrete valuation ring. Therefore all we need to show is that $A_{\mathfrak{m}} \neq 0$. But this is clear from the isomorphism $A \oplus A \cong R \oplus A^2$. ■

3. NOETHERIAN RINGS; BASS'S THEOREMS

Now we turn to possible generalizations of part (1.1) of Steinitz's theorem, the existence theorem, to more general classes of Noetherian rings. It turns out that the "right" generalization is to do away with invertibility of ideals but to preserve the two-generator property. Of course, by Theorem 4, we lose (1.3), but in one important case there is still a canonical form for direct sums of ideals. For reasons discussed in Section 5, we will deal exclusively with reduced rings (those without nonzero nilpotent elements). The first systematic study of the problem was undertaken by Bass in the early sixties, in [1] and [2]. Recall that a module M over a commutative ring R is said to be torsion-free provided $rx \neq 0$ whenever r is a non-zero-divisor of R and x is a nonzero element of M.

THEOREM 5 (Bass, 1962). *Let R be a Noetherian reduced ring such that every finitely generated torsion-free module is isomorphic to a direct sum of ideals. Then R has dimension at most one.*

Since a reduced Noetherian ring of dimension zero is just a direct product of finitely many fields, we ask which one-dimensional Noetherian reduced rings satisfy (1.1). The complete answer is known only for rings satisfying a mild additional hypothesis, which we now describe.

The total quotient ring S of a reduced Noetherian ring R is always zero-dimensional, so it behaves very much like the quotient field of an integral domain. The set $\tilde{R}$ of elements of S that are integral over R is a subring of S, called the integral closure (or normalization) of R. A Noetherian domain D of dimension one is a Dedekind domain if and only if it is integrally closed, i.e., $D = \tilde{D}$. The integral closure of a reduced one-dimensional Noetherian ring is the direct product of a finite number of Dedekind domains. It is not surprising, therefore, that one invariably tries to use the integral closure when proving theorems on direct-sum decompositions of modules over one-dimensional rings. The following hypothesis often enables one to pull decompositions over $\tilde{R}$ back to R:

$(*)$ The normalization $\tilde{R}$ is finitely generated as an R-module.

This hypothesis is satisfied for all the reduced rings one encounters in number theory or algebraic geometry. For example, it holds for any reduced ring that is finitely generated as an algebra over a field or over **Z**. An example of a one-dimensional Noetherian domain not satisfying $(*)$ can be found in the Appendix to Nagata's book [19].

THEOREM 6 (Bass, 1963). *The following are equivalent for a Noetherian domain D satisfying* $(*)$:

(a) *Every finitely torsion-free D-module is isomorphic to a direct sum of ideals of D.*
(b) *Every ideal of D is generated by two elements.*

In the same paper [2] Bass gave several other characterizations of the domains satisfying (a). Geometrically, they correspond to curves whose only singularities are double points (cusps or nodes). He also showed that (b) implies (a) if "domain" is replaced by "reduced ring," but that the converse can fail. The simplest example is the affine coordinate ring of three distinct lines (coplanar or not) meeting in a single point: $k[x,y]/xy(x-y)$ or $k[x,y,z]/[(x,y)\cap(x,z)\cap(y,z)]$. Remarkably, (a) fails for the affine coordinate ring of the union of *four* lines through a point. This follows from a general theorem, published by Dade [8] in 1963, on the existence of big indecomposable modules.

The problem of determining which reduced Noetherian rings satisfy (a) was studied by several authors, including Nazarova and Roiter [20], Greither [11], and Levy and Wiegand [18], and was finally answered [for rings satisfying $(*)$] by Haefner and Levy [15] in 1988. The criteria (a little too technical to state here) are a subtle arithmetic condition on the local rings together with an interesting graph-theoretic condition on the set of prime ideals of the ring.

The one-dimensional reduced rings that satisfy $(*)$ and in which every ideal is generated by two elements are now known as Bass rings. In [18] Levy and Wiegand gave a complete set of invariants for direct sums of ideals over Bass rings. The data determining such a module are the genus (local isomorphism class) and the isomorphism class of the product of the ideals. This result can be viewed as a weak version of Steinitz's theorem. For Bass *domains* there is actually a canonical form, discovered by Borevic and Fadeev [3] in 1966 in a slightly restricted setting, and derived in general in [18]:

THEOREM 7. *Let D be a Bass domain, and let M be a nonzero finitely generated torsion-free D-module. Then there are rings D_i such that $\tilde{D} \supseteq D_m \supseteq \cdots \supseteq D_1 \supseteq D$ and an invertible ideal A of D_m such that $M \cong D_1 \oplus \cdots \oplus D_{m-1} \oplus A$. Moreover, the rings D_i are uniquely determined by M, and A is uniquely determined up to isomorphism.*

Of course, when D is a Dedekind domain, $D = \tilde{D}$, and we recover the canonical form we deduced from Steinitz's theorem.

4. A GENERAL UNIQUENESS THEOREM

In this section we will prove Theorem 3 without the assumption on the existence of non-zero-divisors. (This answers a question raised by Eisenbud [9, p. 136] in 1989.) The tradeoff is that we must *assume* that $m = n$. For example, suppose R has nonzero ideals A and B such that $A \cap B = (0)$, and let $C = A + B$. Then $A \oplus B = C$ but $AB \not\cong C$.

THEOREM 8. *Let A_i, B_i be ideals of the commutative ring R, and suppose the R-modules $A_1 \oplus \cdots \oplus A_m$ and $B_1 \oplus \cdots \oplus B_m$ are isomorphic. Then the ideal products $A_1 \cdots A_m$ and $B_1 \cdots B_m$ are isomorphic R-modules.*

For reduced Noetherian rings, there is a short proof using the fact that $A_1 \cdots A_m$ is the reduction, modulo torsion, of the mth exterior power of $(A_1 \oplus \cdots \oplus A_m)$. A different sort of proof is available for semilocal (Noetherian) rings: one can apply the Krull-Schmidt theorem to prove Theorem 8 for complete local rings, and then use faithfully flat descent. These special cases seem to be fairly well known, but we believe Theorem 8 to be new, even for Noetherian rings.

Our proof depends on a determinant argument much like Kaplansky's. A key ingredient of Kaplansky's proof, however—the fact that homomorphisms between ideals are multiplications by elements in the total quotient ring—is missing. In fact, homomorphisms don't even commute in general. For example, let V be a two-dimensional vector space over a field k, and make $R := k \oplus V$ into a ring with multiplication $(a \oplus x)(b \oplus y) = ab \oplus (ay + bx)$, for

$a, b \in k$ and $x, y \in V$. Then V is a nilpotent ideal, and its R-endomorphism ring is the noncommutative ring $\mathrm{Hom}_k(V, V)$. In order to deal with this problem, we will redefine multiplication of homomorphisms to make it commutative.

LEMMA. *Let $f: A \to B$ and $g: C \to D$ be R-homomorphisms between ideals of the commutative ring R. Then there is a unique R-homomorphism $h: AC \to BD$ satisfying $h(ac) = f(a)g(c)$ for all $a \in A$ and $c \in C$.*

Proof. Uniqueness is clear, since the elements of the form ac generate AC. For the existence, note that f carries the R-submodule AC of A into BC, so let $f_C: AC \to BC$ be the map induced by f. Define $g_B: BC \to BD$ similarly, and let $h = g_B \circ f_C$. ■

We denote the map h simply by fg. We take the category-theoretic point of view that every map has a well-defined domain and target. Thus we have a commutative monoid structure on the set of maps between pairs of ideals of R. Furthermore, the distributive law $f(g + h) = fg + fh$ holds whenever g and h have the same domain and target.

Now let $f: A_1 \oplus \cdots \oplus A_m \to B_1 \oplus \cdots \oplus B_m$ be an R-homomorphism, and let $[f_{ij}: A_j \to B_i]$ be the matrix of f. For every permutation $\sigma \in S_m$, the product $\prod_{i=1}^{m} f_{i,\sigma(i)}$ is an R-homomorphism from $A_1 \cdots A_m$ to $B_1 \cdots B_m$, and we define an R-homomorphism $\det f: A_1 \cdots A_m \to B_1 \cdots B_m$ by the usual formula

$$\det f = \sum_{\sigma \in S_m} |\sigma| \prod_{i=1}^{m} f_{i,\sigma(i)}. \tag{4.1}$$

If f is an isomorphism, we can do all these things to the inverse map g. Since the determinant of the identity map on a direct sum of ideals is clearly the identity map on their product, it will suffice to prove the following product rule for determinants:

THEOREM 9. *Let $f: A_1 \oplus \cdots \oplus A_m \to B_1 \oplus \cdots \oplus B_m$ and $g: B_1 \oplus \cdots \oplus B_m \to C_1 \oplus \cdots \oplus C_m$ be R-homomorphisms between direct sums of ideals of R. Then $\det(g \circ f) = (\det g) \circ (\det f)$, as maps from $A_1 \cdots A_m$ to $C_1 \cdots C_m$.*

It is tempting to say that this follows immediately from the formal properties of determinants. Unfortunately, however, there are two very different products involved—composition, and the funny product defined by the Lemma. In fact, the funny product gf doesn't even make sense, and

$(\det g)(\det f)$ maps $A_1 \cdots A_m B_1 \cdots B_m$ to $B_1 \cdots B_m C_1 \cdots C_m$, which is not at all what we want. While it is possible to prove Theorem 9 directly, it seems notationally cleaner to work first with one map and one element, instead of with two maps.

Let $x = [x_{jk}]$ be a $m \times m$ matrix whose jth row consists of elements of the ideal A_j. The kth column of x is then an element of $A_1 \oplus \cdots \oplus A_m$. Further, $\det x \in A_1 \cdots A_m$, since each term in the usual expansion

$$\det x = \sum_{\sigma \in S_m} |\sigma| \prod_{j=1}^{m} x_{j,\sigma(j)} \tag{4.2}$$

is in $A_1 \cdots A_m$.

Now fx is an $m \times m$ matrix whose ith row consists of elements of B_i. Of course, the kth column of fx is the image, under the map f, of the kth column of x. Therefore we can define $\det fx$ exactly as in (4.2), and we claim

$$\det fx = (\det f)[\det x], \tag{4.3}$$

where the square brackets indicate that we apply the function $\det f$ to the ring element $\det x$.

We will prove this by adapting the old-fashioned proof (devoid of row echelon form) of the usual product rule for determinants. (See, for example, Elbert Walker's abstract-algebra book [28].) The only problem is to avoid any occurrence of terms $f_{ij}x_{kl}$ with $j \neq k$, and we will need to invoke the Lemma to accomplish this.

To prove (4.3), we have $\det fx = \det[\Sigma_j f_{ij} x_{jk}] = \Sigma_\sigma |\sigma| \Pi_i (\Sigma_j f_{ij} x_{j,\sigma(i)}) = \Sigma_\sigma |\sigma| \Sigma_\theta (\Pi_i f_{i,\theta(i)} x_{\theta(i),\sigma(i)})$, where θ ranges over all *functions* from $\{1, \ldots, m\}$ to $\{1, \ldots, m\}$. We claim that

$$\sum_{\sigma \in S_m} |\sigma| \prod_i f_{i,\theta(i)} x_{\theta(i),\sigma(i)} = 0 \qquad \text{if } \theta \text{ is not a permutation.} \tag{4.4}$$

Following Walker's argument, we choose $s \neq t$ such that $\theta(s) = \theta(t)$ and let π be the transposition (st). Notice that

$$\left(f_{s,\theta(s)} x_{\theta(s),\sigma(s)}\right)\left(f_{t,\theta(t)} x_{\theta(t),\sigma(t)}\right) = \left(f_{s,\theta(s)} x_{\theta(s),\sigma\pi(s)}\right)\left(f_{t,\theta(t)} x_{\theta(t),\sigma\pi(t)}\right),$$

since [with $\theta(s) = \theta(t) = k$] both sides are equal to $(f_{s,k} f_{t,k})(x_{k,\sigma(s)} x_{k,\sigma(t)})$, where $f_{s,k} f_{t,k} : A_k^2 \to B_s B_t$ is the map given by the Lemma. Therefore $\Pi_i f_{i,\theta(i)} x_{\theta(i),\sigma(i)} = \Pi_i f_{i,\theta(i)} x_{\theta(i),\sigma\pi(i)}$, since the ith factors of the two products are identical for $i \notin \{s, t\}$. Now (4.4) follows, since the σth and $(\sigma\pi)$th terms cancel each other in pairs.

Returning to our calculation of $\det fx$, we see that the sum over all functions θ can be replaced by the sum over all permutations $\tau \in S_m$. Then $\det fx = \Sigma_\sigma |\sigma| \Sigma_\tau (\Pi_i f_{i,\tau(i)} x_{\tau(i),\sigma(i)})$. Now put $\rho = \tau^{-1}\sigma$, to get

$$\begin{aligned}\det fx &= \sum_\tau |\tau| \sum_\rho |\rho| \Big(\prod_i f_{i,\tau(i)} x_{\tau(i),\tau\rho(i)} \Big) \\ &= \sum_\tau |\tau| \sum_\rho |\rho| \Big(\prod_i f_{i,\tau(i)} \Big) \Big(\prod_i x_{\tau(i),\tau\rho(i)} \Big) \\ &= \sum_\tau |\tau| \sum_\rho |\rho| \Big(\prod_i f_{i,\tau(i)} \Big) \Big(\prod_i x_{i,\rho(i)} \Big) = (\det f)[\det x],\end{aligned}$$

and (4.3) is proved.

Now, to prove Theorem 9, let x be an $m \times m$ matrix as in (4.2). Using (4.2) three times, we have $(\det(g \circ f))[\det x] = \det((g \circ f)x) = \det(g(fx)) = (\det g)[\det fx] = (\det g)[(\det f)[\det x]] = ((\det g)\circ(\det f))[\det x]$. Since $A_1 \cdots A_m$ is generated by determinants of these elements x (in fact, by the determinants of the diagonal x's), Theorem 9 is proved.

5. GENERALIZATIONS AND QUESTIONS

There are many directions in which one might try to expand the scope of the results we have surveyed, particularly the results on the existence of decompositions in Section 3. For example, Bass's theorem [1] says that a quasilocal (i.e., local but not necessarily Noetherian) domain having an ideal requiring three generators always has a finitely generated torsion-free module that is not isomorphic to a direct sum of ideals. We ask whether the converse holds:

QUESTION 1. Let R be a quasilocal domain in which each finitely generated ideal is generated by two elements. Is every finitely generated torsion-free R-module isomorphic to a direct sum of ideals?

David Rush [23] has a nice technique for circumventing the hypothesis $(*)$ in certain sorts of problems. It seems likely that one could use his methods to answer the following question affirmatively:

QUESTION 2. Is Theorem 6 still valid if $(*)$ is deleted from the hypotheses?

It would be nice to salvage some of the results of Section 3 for Noetherian rings with nonzero nilpotent elements. However, as soon as R has embedded primes (nonminimal primes consisting of zero divisors), there is no hope of decomposing all finitely generated torsion-free modules. (See the first paragraph of Section 3 for the definition of "torsion-free.") In fact, we will see that such a ring always has torsion-free modules that cannot even be embedded in free modules. Following Bass, we say a module M is "torsionless" provided the canonical map $M \to M^{**}$ is injective, where X^* denotes the dual $\mathrm{Hom}_R(X, R)$.

REMARK. Let M be a finitely generated module over a Noetherian ring R. Then M is torsionless if and only if M can be embedded in a free module.

Proof. If X is any finitely generated R-module, we claim that X^* embeds in a free module. To see this let $F \to X$ be a surjection, with F finitely generated and free. The dual map $X^* \to F^*$ is then injective, and F^* is free. Taking $X = M^*$, we see that if M is torsionless, then it embeds in a free module.

Conversely, if $M \to F$ is an embedding with F free, we may assume F has finite rank. There is a commutative diagram:

$$\begin{array}{ccc} M & \to & F \\ \downarrow & & \downarrow \\ M^{**} & \to & F^{**} \end{array}$$

Since $M \to F$ is injective and $F \to F^{**}$ is an isomorphism, the composite map $M \to F^{**}$ is injective. Therefore $M \to M^{**}$ is injective. ∎

THEOREM 10. *The following conditions on the Noetherian ring R are equivalent*:

(1) *Every finitely generated torsion-free R-module is isomorphic to a submodule of a free R-module.*

(2) *Every finitely generated torsion-free R-module is torsionless.*

(3) *R has no embedded primes, and R_P is Gorenstein for every minimal prime ideal P of R.*

Proof. (1) and (2) are equivalent by the remark, and the equivalence of (2) and (3) was established by Vasconcelos in [27]. ∎

The theorem suggests that for general Noetherian rings we should forget about torsion-free modules and look instead for criteria for every finitely generated submodule of a free module to be isomorphic to a direct sum of ideals.

QUESTION 3. Let R be a Noetherian ring in which every ideal is generated by two elements. Is every finitely generated submodule of a free R-module isomorphic to a direct sum of ideals of R?

We close by listing explicitly a question implicit in Section 2.

QUESTION 4. Let R be a Prüfer domain in which every finitely generated ideal is generated by two elements. Does R have the Steinitz property (SP)? In particular, does the ring of integer-valued polynomials have the Steinitz property?

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